A Noncommutative Analogue of $|D(X^k)| = |kX^{k-1}|$

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ABSTRACT

The paper concerns alternating powers of a Hilbert space. Let $\wedge^k$ be defined by $\wedge^k(A)(x_1 \wedge \cdots \wedge x_k) = Ax_1 \wedge \cdots \wedge Ax_k$. It is proved that the norm of the linear map $D\wedge^k(A)$ depends only upon $|A|$ and is assumed at the identity.

INTRODUCTION

If $\mathcal{H}$ is a (complex) Hilbert space, let $\wedge^k: \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\wedge^k\mathcal{H})$ be the map defined by $\wedge^k(A)(x_1 \wedge \cdots \wedge x_k) = Ax_1 \wedge \cdots \wedge Ax_k$. Then $\wedge^k$ is a differentiable map and $D\wedge^k(A): \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\wedge^k\mathcal{H})$ is a bounded operator. The main result of this paper is:

THEOREM $(*)$. $\|D\wedge^k(A)\| = \|(D\wedge^k(|A|))(1_{\mathcal{H}})\|$, *where* $|A| = (A^*A)^{1/2}$.

The title of the paper derives from the fact that for any T in $\mathcal{L}(\mathcal{H})$,

$$\begin{aligned}(D\wedge^k(T))(1_{\mathcal{H}})(x_1 \wedge \cdots \wedge x_k) = {} & x_1 \wedge Tx_2 \wedge \cdots \wedge Tx_k \\ & + Tx_1 \wedge x_2 \wedge Tx_3 \wedge \cdots \wedge Tx_k \\ & + \cdots + Tx_1 \wedge \cdots \wedge Tx_{k-1} \wedge x_k,\end{aligned}$$

so that $D\wedge^k(T)(1)$ is a "noncommutative analogue of kT^{k-1}." The following corollary reveals the above result in a clearer form.

COROLLARY. *If A is a compact operator (in particular, if $\mathcal{H}$ is finite-dimensional), and if $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_k$ are the k largest singular values of A*

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(i.e., eigenvalues of $|A|$*) counting multiplicity, then*

$$\|D\wedge^k(A)\| = s_{k-1}(\alpha_1, \alpha_2, \ldots, \alpha_k),$$

where $s_{k-1}(\alpha_1,\ldots,\alpha_k)$ *is the* $(k-1)$*th elementary symmetric polynomial of* $\alpha_1,\ldots,\alpha_k$.

The statement of the corollary remains valid even for noncompact A, if the phrase "k largest singular values of A" is interpreted suitably.

PRELIMINARIES

The symbol $\mathcal{H}$ will always denote a complex Hilbert space, and $\mathcal{L}(\mathcal{H})$ the C^*-algebra of all bounded operators on $\mathcal{H}$. The kth alternating and tensor powers of $\mathcal{H}$ (with their natural Hilbert-space structures) will be denoted by $\wedge^k\mathcal{H}$ and $\otimes^k\mathcal{H}$ respectively ($k>1$). If $A, A_1,\ldots,A_k \in \mathcal{L}(\mathcal{H})$, then $A\wedge\cdots\wedge A$ and $A_1\otimes\cdots\otimes A_k$ will denote the (uniquely defined) operators on $\wedge^k\mathcal{H}$ and $\otimes^k\mathcal{H}$ respectively, satisfying

$$(A\wedge\cdots\wedge A)(x_1\wedge\cdots\wedge x_k) = Ax_1\wedge\cdots\wedge Ax_k$$

and

$$(A_1\otimes\cdots\otimes A_k)(x_1\otimes\cdots\otimes x_k) = A_1x_1\otimes\cdots\otimes A_kx_k$$

for all $x_1,\ldots,x_k$ in $\mathcal{H}$. (Note that $A\wedge\cdots\wedge A$ is *not* the wedge product in $\wedge^k\mathcal{L}(\mathcal{H})$; in fact, $A\wedge A$ will not be zero unless A has rank at most one.) We shall write $\wedge^k A = A\wedge\cdots\wedge A$ and $\otimes^k A = A\otimes\cdots\otimes A$. Define $\varphi_A: \mathcal{L}(\mathcal{H})\to\mathcal{L}(\wedge^k\mathcal{H})$ and $\psi_A: \mathcal{L}(\mathcal{H})\to\mathcal{L}(\otimes^k\mathcal{H})$ as follows:

$$\varphi_A(B)(x_1\wedge\cdots\wedge x_k) = Bx_1\wedge Ax_2\wedge\cdots\wedge Ax_k + Ax_1\wedge Bx_2\wedge Ax_3\wedge\cdots\wedge Ax_k + \cdots + Ax_1\wedge\cdots\wedge Ax_{k-1}\wedge Bx_k \tag{1}$$

whenever $x_1,\ldots,x_k\in\mathcal{H}$; and

$$\psi_A(B) = B\otimes A\otimes A\otimes\cdots\otimes A + A\otimes B\otimes A\otimes\cdots\otimes A + \cdots + A\otimes A\otimes\cdots\otimes A\otimes B \tag{2}$$

[*Note:* In general, if $A_1, \ldots, A_k \in \mathcal{L}(\mathcal{H})$, then $x_1 \wedge \cdots \wedge x_k \mapsto A_1x_1 \wedge \cdots \wedge A_kx_k$ is not a meaningfully defined operator; nevertheless, (1) does define a well-defined operator which we shall sometimes denote by $B \wedge A \wedge \cdots \wedge A + \cdots + A \wedge \cdots \wedge A \wedge B$, although individual terms of this sum do not make sense.]

Now, $\wedge^k : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\wedge^k \mathcal{H})$ is a differential map of Banach spaces. So, if $A \in \mathcal{L}(\mathcal{H})$, then $D \wedge^k(A) : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\wedge^k \mathcal{H})$ is a bounded operator. In fact, it can be shown (cf. [1]) that $D \wedge^k(A)(B) = \varphi_A(B)$ for all $A, B \in \mathcal{L}(\mathcal{H})$; i.e., $D \wedge^k(A) = \varphi_A$, as an operator from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\wedge^k \mathcal{H})$.

Finally, if $A \in \mathcal{L}(\mathcal{H})$, we shall write $|A|$ for the positive square root of A^*A; i.e., $|A| = (A^*A)^{1/2}$.

Proof of Theorem $(*)$. In view of the foregoing remarks, it is to be proved that $\|\varphi_A\| = \|\varphi_{|A|}(1)\|$ for every A in $\mathcal{L}(\mathcal{H})$. (Here, and elsewhere, the identity operator on $\mathcal{H}$ will be denoted by 1.) The equality asserted above is a consequence of the following theorems, whose proofs are accomplished by a series of lemmas.

THEOREM 1. *If* $A \geqslant 0$, *then* φ_A *is a completely positive map, and hence* $\|\varphi_A\| = \|\varphi_A(1)\|$.

THEOREM 2. *If* $A \in \mathcal{L}(\mathcal{H})$, *then*

$$\|\varphi_A\| = \|\varphi_{|A|}\| = \|\varphi_{|A^*|}\| = \|\varphi_{A^*}\|.$$

Before getting into the proof of Theorem 1, let us recall some facts about completely positive maps (hereafter abbreviated to c.p. maps). A linear map $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is said to be a c.p. map if, for every positive integer n, the $n \times n$ matrix $(\varphi(B_{ij}))$ defines a positive operator on $\mathcal{K}^{(n)} = \mathcal{K} \oplus \cdots \oplus \mathcal{K}$, whenever (B_{ij}) defines a positive operator on $\mathcal{H}^{(n)}$. A theorem of Stinespring's (cf. [2]) states that $\varphi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ is a c.p. map iff there exists a representation $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}_\pi)$ and a bounded operator $T : \mathcal{K} \to \mathcal{H}_\pi$ such that $\varphi(B) = T^*\pi(B)T \ \forall B \in \mathcal{L}(\mathcal{H})$. Since representations have norm one, it follows that c.p. maps attain their norm at the identity: i.e., $\|\varphi\| = \|\varphi(1)\|$ for every c.p. map φ.

LEMMA 1.1. *If* $A \geqslant 0$, *then* $B \mapsto A \otimes B$ *is a c.p. map (from* $\mathcal{L}(\mathcal{H})$ *to* $\mathcal{L}(\mathcal{H} \otimes \mathcal{H})$*).*

Proof. Suppose $(B_{ij})_{1\leq i,j\leq n}$ defines a positive operator on $\mathcal{H}^{(n)}$. Then, we must check that $(A\otimes B_{ij})_{1\leq i,j\leq n}$ defines a positive operator on $(\mathcal{H}\otimes\mathcal{H})^{(n)}$. Under the natural identification of $(\mathcal{H}\otimes\mathcal{H})^{(n)}$ with $\mathcal{H}\otimes\mathcal{H}^{(n)}$, it is easy to see that $(A\otimes B_{ij})_{1\leq i,j\leq n}$ corresponds to $A\otimes(B_{ij})_{1\leq i,j\leq n}$, which is positive, since the tensor product of positive operators is positive. ■

In the following lemma, S_k denotes the symmetric group on k objects. For σ in S_k, let ε_σ denote the signature of the permutation σ, and let U_σ denote the unitary operator on $\otimes^k\mathcal{H}$ satisfying $U_\sigma(x_1\otimes\cdots\otimes x_k)=x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(k)}$ $\forall x_1,\ldots,x_k$ in $\mathcal{H}$. Let $V\colon\wedge^k\mathcal{H}\to\otimes^k\mathcal{H}$ be the natural inclusion map, defined by

$$V(x_1\wedge\cdots\wedge x_k)=\left(\frac{1}{k!}\sum_{\sigma\in S_k}\varepsilon_\sigma U_\sigma\right)(x_1\otimes\cdots\otimes x_k)\qquad \forall x_1,\ldots,x_k\in\mathcal{H}. \quad (3)$$

LEMMA 1.2. *If* φ_A, ψ_A, *and* V *are defined as in Equations* (1), (2), *and* (3), *then*

$$\varphi_A(B)=V^*\psi_A(B)V\qquad \forall A,B\in\mathcal{L}(\mathcal{H})$$

Proof. It is easily seen that V is isometric. Let $P\in\mathcal{L}(\otimes^k\mathcal{H})$ be the projection onto the range of V. It follows from the definitions that

$$P=\frac{1}{k!}\sum_{\sigma\in S_k}\varepsilon_\sigma U_\sigma. \quad (4)$$

If $A,B\in\mathcal{L}(\mathcal{H})$ and if $x_1,\ldots,x_k\in\mathcal{H}$, then

$$\begin{aligned}
&V\varphi_A(B)V^*(x_1\otimes\cdots\otimes x_k)\\
&\quad=V\varphi_A(B)(x_1\wedge\cdots\wedge x_k)\\
&\quad=V(Bx_1\wedge Ax_2\wedge\cdots\wedge Ax_k+\cdots+Ax_1\wedge\cdots\wedge Ax_{k-1}\wedge Bx_k)\\
&\quad=\left(\frac{1}{k!}\sum_{\sigma\in S_k}\varepsilon_\sigma U_\sigma\right)(Bx_1\otimes\cdots\otimes Ax_k+\cdots+Ax_1\otimes\cdots\otimes Bx_k)\\
&\quad=P\psi_A(B)(x_1\otimes\cdots\otimes x_k),
\end{aligned}$$

and hence

$$V\varphi_A(B)V^* = P\psi_A(B). \tag{5}$$

Another routine check shows that for all σ in S_k,

$$U_\sigma(A_1\otimes \cdots \otimes A_k)U_{\sigma^{-1}} = A_{\sigma(1)}\otimes \cdots \otimes A_{\sigma(k)}$$

for all $A_1,\dots,A_k$ in $\mathcal{L}(\mathcal{H})$. This, and the symmetry of $\psi_A(B)$, imply that

$$U_\sigma\psi_A(B)U_{\sigma^{-1}} = \psi_A(B).$$

In other words, $\psi_A(B)$ commutes with every U_σ, and hence [by (4)] $\psi_A(B)$ commutes with the projection P. So Equation (5) may be rewritten as

$$V\varphi_A(B)V^* = P\psi_A(B) = P\psi_A(B)P.$$

Because $V^*V=1$ and $VV^*=P$, this gives the announced conclusion. ∎

Proof of Theorem 1. Let $A\geqslant 0$. By Lemma 1.1 and Stinespring's result, there exists a representation $\pi:\mathcal{L}(\mathcal{H})\to\mathcal{L}(\mathcal{H}_\pi)$ and a bounded operator $T:\otimes^k\mathcal{H}\to\mathcal{H}_\pi$ such that $\psi_A(B)=T^*\pi(B)T$ $\forall B\in\mathcal{L}(\mathcal{H})$. Lemma 1.2 then implies that $\varphi_A(B)=(TV)^*\pi(B)(TV)$, thereby proving that φ_A is c.p. ∎

Although the index is normally defined only for semi-Fredholm operators, we shall define, for every A in $\mathcal{L}(\mathcal{H})$,

$$\operatorname{ind}(A) = \dim(\ker A) - \dim(\ker A^*),$$

(with the understanding that $\infty-\infty=0$). Recall that the polar decomposition theorem expresses A as a product: $A=U|A|$, where U is a partial isometry which maps $\ker A$ to zero and $\ker^{\perp}A$ isometrically onto $\ker^{\perp}A^*$. According to whether $\operatorname{ind}A\geqslant 0$ or $\operatorname{ind}A\leqslant 0$, we may redefine U on $\ker A$ so that U is coisometric (i.e. $UU^*=1$) or U is isometric. The modified operator U still satisfies $A=U|A|$, $U^*A=|A|$, $U^*U|A|=|A|$. This observation will be used in the proof of the next two lemmas.

LEMMA 2.1. *Let $A\in\mathcal{L}(\mathcal{H})$. Then,*

(i) *if* $\operatorname{ind}A\geqslant 0$,

$$\|\varphi_{|A^*|}\| \leqslant \|\varphi_A\| \leqslant \|\varphi_{|A|}\|;$$

(ii) *if* ind $A \leqslant 0$,

$$\|\varphi_{|A^*|}\| \geqslant \|\varphi_A\| \geqslant \|\varphi_{|A|}\|.$$

Proof. (i): Here ind $A \geqslant 0$ and ind $A^* \leqslant 0$. So, by the observation made earlier, there exist a coisometry U and an isometry W such that

$$A = U|A|, \qquad U^*A = |A|,$$

$$A^* = W|A^*|, \qquad W^*A^* = |A^*|.$$

Then, for any B in $\mathfrak{L}(\mathfrak{H})$, since $(\wedge^k U)(\wedge^k U^*) = 1$,

$$\begin{aligned}
\|\varphi_A(B)\| &= \|B \wedge \cdots \wedge A + \cdots + A \wedge \cdots \wedge B\| \\
&= \|(\wedge^k U)(\wedge^k U^*)(B \wedge \cdots \wedge A + \cdots + A \wedge \cdots \wedge B)\| \\
&\leqslant \|(\wedge^k U^*)(B \wedge \cdots \wedge A + \cdots + A \wedge \cdots \wedge B)\| \\
&= \|U^*B \wedge \cdots \wedge U^*A + \cdots + U^*A \wedge \cdots \wedge U^*A \wedge U^*B\| \\
&= \|\varphi_{U^*A}(U^*B)\| \\
&= \|\varphi_{|A|}(U^*B)\| \\
&\leqslant \|\varphi_{|A|}\|\|B\| \qquad (\text{since } \|U^*\| = 1),
\end{aligned}$$

and hence,

$$\|\varphi_A\| \leqslant \|\varphi_{|A|}\|.$$

On the other hand, since $|A^*| \geqslant 0$, we have, by Theorem 1,

$$\begin{aligned}
\|\varphi_{|A^*|}\| &= \|1 \wedge \cdots \wedge |A^*| + \cdots + |A^*| \wedge \cdots \wedge 1\| \\
&= \|(1 \wedge \cdots \wedge |A^*| + \cdots + |A^*| \wedge \cdots \wedge 1)(\wedge^k W^*)(\wedge^k W)\| \\
&\qquad\qquad (\text{since } \wedge^k W \text{ is isometric})
\end{aligned}$$

$$\leqslant \left\|\left(1\wedge\cdots\wedge|A^*|+\cdots+|A^*|\wedge\cdots\wedge 1\right)\wedge^k W^*\right\|$$

$$=\left\|W^*\wedge\cdots\wedge|A^*|W^*+\cdots+|A^*|W^*\wedge\cdots\wedge W^*\right\|$$

$$=\left\|\varphi_{|A^*|W^*}(W^*)\right\|$$

$$=\|\varphi_A(W^*)\| \qquad (\text{since}\quad A=|A^*|W^*)$$

$$\leqslant\|\varphi_A\| \qquad (\text{since}\quad \|W^*\|=1).$$

So case (i) is proved. The proof of case (ii) is entirely analogous. ■

LEMMA 2.2. *Let* $A\in\mathcal{L}(\mathcal{H})$, *and suppose* $\operatorname{ind} A\leqslant 0$. *Then* $|A^*|$ *is unitarily equivalent to* $|A|\oplus 0$, *where* 0 *is the zero operator on a space of dimension* $-\operatorname{ind} A$.

Proof. This follows from the comments made above on the polar decomposition. ■

Henceforth, $s_{k-1}(\lambda_1,\dots,\lambda_k)$ will denote the $(k-1)$st elementary symmetric polynomial of the λ_i's; thus,

$$s_{k-1}(\lambda_1,\dots,\lambda_k)=\lambda_2\lambda_3\cdots\lambda_k+\lambda_1\lambda_3\cdots\lambda_k+\cdots+\lambda_1\lambda_2\cdots\lambda_{k-1}.$$

LEMMA 2.3. *If* $P\geqslant 0$, *then* $\|\varphi_P\|=\|\varphi_{P\oplus 0}\|$.

Proof.

Case (*i*): *P has pure point spectrum.* Thus, there is an orthonormal basis $\{e_i\}$ of $\mathcal{H}$ and numbers $\lambda_i\geqslant 0$ such that $Pe_i=\lambda_i e_i\ \forall i$. Then, clearly, for distinct indices $i_1,\dots,i_k$,

$$\varphi_P(1)\left(e_{i_1}\wedge\cdots\wedge e_{i_k}\right)=s_{k-1}\left(\lambda_{i_1},\dots,\lambda_{i_k}\right)e_{i_1}\wedge\cdots\wedge e_{i_k}.$$

Thus, $\varphi_P(1)$ is diagonalizable with respect to the basis $\{e_{i_1}\wedge\cdots\wedge e_{i_k}\}$ for $\wedge^k\mathcal{H}$. Hence,

$$\|\varphi_P(1)\|=\sup\left\{s_{k-1}\left(\lambda_{i_1},\dots,\lambda_{i_k}\right): i_j\text{'s distinct}\right\}. \tag{6}$$

Since the eigenvalues of $P\oplus 0$ are the λ_i's together with an appropriate number of zeros, and since a formula analogous to (6) holds with $P\oplus 0$ in place of P, and since the λ_i's are nonnegative it is clear that

$$\|\varphi_P\| = \|\varphi_P(1)\| = \|\varphi_{P\oplus 0}(1)\| = \|\varphi_{P\oplus 0}\|.$$

Case (ii): *P arbitrary.* Since positive operators with pure point spectrum are dense in the family of all positive operators, and since $\|\varphi_P\|$ and $\|\varphi_{P\oplus 0}\|$ both vary continuously with P, the proof is complete. ■

Proof of Theorem 2. Lemmas 2.2 and 2.3 imply that $\|\varphi_{|A|}\| = \|\varphi_{|A^*|}\|$ for any $A\in\mathcal{L}(\mathcal{H})$. (If ind $A\geqslant 0$, apply the lemmas to A^* in place of A.) This, together with Lemma 2.1, now shows that $\|\varphi_A\| = \|\varphi_{|A|}\| = \|\varphi_{|A^*|}\|$. Interchanging A and A^*, we see that the common value of the three expressions above agrees with $\|\varphi_{A^*}\|$, thus completing the proof of Theorem 2. ■

Corollary. *If A is a compact operator on $\mathcal{H}$ (in particular, if $\mathcal{H}$ is finite-dimensional and A is arbitrary), and if $\alpha_1\geqslant\alpha_2\geqslant\cdots\downarrow 0$ is an enumeration of the eigenvalues of $|A|$, counted with multiplicity, then*

$$\|D\wedge^k(A)\| = s_{k-1}(\alpha_1,\dots,\alpha_k). \tag{7}$$

Proof. $|A|$ is a positive operator with pure point spectrum. It has eigenvalues $\alpha_1\geqslant\alpha_2\geqslant\cdots\downarrow 0$. The assertion follows from Equation (6) in the proof of Lemma 2.3. ■

Remarks.

(1) Equation (7) in the statement of the Corollary continues to hold for noncompact A as well, if the "singular values" of A are appropriately interpreted. Thus, if points in the essential spectrum of A are assigned infinite multiplicity, and if $\alpha_1,\dots,\alpha_k$ are the k "largest" points in sp$(|A|)$, then $\|D\wedge^k(A)\| = s_{k-1}(\alpha_1,\dots,\alpha_k)$.

(2) A result similar to Theorem 1 can be stated with Schur products in place of alternating products. More precisely, represent $\mathcal{L}(\mathcal{H})$ concretely as matrices, and let $A\cdot B$ be the Schur (or Hadamard) product of A and B. [If $A=(\alpha_{ij})$ and $B=(\beta_{ij})$, then $A\cdot B=(\alpha_{ij}\beta_{ij})$.] Define $\Sigma^k\colon\mathcal{L}(\mathcal{H})\to\mathcal{L}(\mathcal{H})$ by

$$\Sigma^k(A) = \underbrace{A\cdot A\cdot\;\cdots\;\cdot A}_{k\text{ terms}}.$$

Then it is easily shown that

$$D\Sigma^k(A)(B) = B\cdot A\cdot \cdots \cdot A + A\cdot B\cdot \cdots \cdot A + \cdots + A\cdot A\cdot \cdots \cdot A\cdot B. \quad (8)$$

However, the Schur product is a compression of the tensor product. More precisely, there exists an isometry $V\colon \mathcal{H} \to \otimes^k \mathcal{H}$ such that

$$A_1\cdot \cdots \cdot A_k = V^*(A_1 \otimes \cdots \otimes A_k)V \quad \forall A_1,\ldots,A_k \in \mathcal{L}(\mathcal{H}).$$

Hence, $D\Sigma^k(A)(B) = V^*\psi_A(B)V$ $\forall A, B \in \mathcal{L}(\mathcal{H})$ [see (2) for the definition of ψ_A]. If $A \geq 0$, then the complete positivity of ψ_A (by Lemma 1.1) implies, as before, that $D\Sigma^k(A)$ is a c.p. map. Hence,

$$\|D\Sigma^k(A)\| = \|D\Sigma^k(A)(1)\|.$$

So, if $A = (\alpha_{ij})$, it follows from Equation (8) that

$$\|D\Sigma^k(A)\| = \|\operatorname{diag}(k\alpha_{ii}^{k-1})\| = k \sup_i \alpha_{ii}^{k-1}.$$

In [1], the authors prove the equality of $\|D\wedge^k(A)\|$ *and* $s_{k-1}(\alpha_1,\ldots,\alpha_k)$ *in the finite-dimensional case. Their proof involves some complicated combinatorics, and is not very clear at certain points. The proof presented here has the advantage of being free of combinatorics, and of being applicable in the infinite-dimensional case as well. However, there are some common points in the two proofs—notably, the reduction to the case where A is positive.* *The author is indebted to Rajendra Bhatia, one of the coauthors of [1], for several fruitful conversations with him, which resulted in this paper.*

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